ALGEBRA (TEAM)

Problem 1. Let $G = \mathrm{GL}_2(\mathbb{C})$.

(1) Prove all finite dimensional representations of G over $\mathbb{C}$ are completely reducible.
(2) Find all irreducible finite dimensional representations of G over $\mathbb{C}$.

Problem 2. Let K and L over $\mathbb{Q}$ be field extensions of prime degrees. Show that if $[KL : \mathbb{Q}] < [K : \mathbb{Q}][L : \mathbb{Q}]$, then the Galois closure of $K/\mathbb{Q}$ equals to the Galois closure of $L/\mathbb{Q}$.

Problem 3. Let G be a finite group. Let N be a minimal nontrivial normal subgroup of G (i.e., N does not properly contain any other nontrivial normal subgroup of G). Show that N is isomorphic to a direct product $L \times ... \times L$ of copies of a single simple group L.